

A note on generically stable measures and *fs* groups

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January 20, 2013

Abstract

We prove (Proposition 2.1) that if μ is a generically stable measure in an *NIP* theory, and $\mu(\phi(x, b)) = 0$ for all b then for some n , $\mu^{(n)}(\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$. As a consequence we show (Proposition 3.2) that if G is a definable group with *fs* in an *NIP* theory, and X is a definable subset of G then X is generic if and only if every translate of X does not fork over \emptyset , precisely as in stable groups, answering positively Problem 5.5 from [3].

1 Introduction and preliminaries

This short paper is a contribution to the generalization of stability theory and stable group theory to *NIP* theories, and also provides another example where we need to resort to measures to prove statements (about definable sets and/or types) which do not explicitly mention measures. The observations in the current paper can and will be used in the future to sharpen existing results around measure and *NIP* theories (and this is why we wanted to record the

*Supported by ISF

†Supported by EPSRC grants EP/F009712/1 and EP/I002294/1

observations here). Included in these sharpenings will be: (i) replacing average types by generically stable types in a characterization of strong dependence in terms of measure and weight in [6], and (ii) showing the existence of “external generic types” (in the sense of Newelski [5]), over any model, for *fsg* groups in *NIP* theories, improving on Lemma 4.14 and related results from [5].

If $p(x) \in S(A)$ is a stationary type in a stable theory and $\phi(x, b)$ any formula, then we know that $\phi(x, b) \in p|_{\mathfrak{C}}$ if and only if $\models \bigwedge_{i=1, \dots, n} \phi(a_i, b)$ for some independent realizations a_1, \dots, a_n of p (for some n depending on $\phi(x, y)$). Hence $\phi(x, b) \notin p|_{\mathfrak{C}}$ for all b implies that (and is clearly implied by) the inconsistency of $\bigwedge_{i=1, \dots, n} \phi(a_i, y)$ for some (any) independent set a_1, \dots, a_n of realizations of p . This also holds for generically stable types in *NIP* theories (as well as for generically stable types in arbitrary theories, with definition as in [7]). In [6], an analogous result was proved for “average measures” in strongly dependent theories. Here we prove it (Proposition 2.1) for generically stable measures in arbitrary *NIP* theories, as well as giving a generalization (Remark 2.2).

The *fsg* condition on a definable group G is a kind of “definable compactness” assumption, and in fact means precisely this in *o*-minimal theories and suitable theories of valued fields (and of course stable groups are *fsg*). Genericity of a definable subset X of G means that finitely many translates of X cover G . Proposition 2.1 is used to show that for X a definable subset of an *fsg* group G , X is generic if and only if every translate of X does not fork over \emptyset . This is a somewhat striking extension of stable group theory to the *NIP* environment.

We work with an *NIP* theory T and inside some monster model \mathfrak{C} . If A is any set of parameters, let $L_x(A)$ denote the Boolean algebra of A -definable sets in the variable x . A *Keisler measure* over A is a finitely additive probability measure on $L_x(A)$. Equivalently, it is a regular Borel probability measure on the compact space $S_x(A)$. We will denote by $\mathfrak{M}_x(A)$ the space of Keisler measures over A in the variable x . We might omit x when it is not needed or when it is included in the notation of the measure itself (*e.g.* μ_x). If X is a sort, or more generally definable set, we may also use notation such $L_X(A)$, $S_X(A)$, $\mathfrak{M}_X(A)$, where for example $S_X(A)$ denote the complete types over A which contain the formula defining X (or which “concentrate on X ”).

Definition 1.1. A type $p \in S_x(A)$ is *weakly random* for μ_x if $\mu(\phi(x)) > 0$ for any $\phi(x) \in L(A)$ such that $p \vdash \phi(x)$. A point b is weakly random for μ over A if $\text{tp}(b/A)$ is weakly random for μ .

We briefly recall some definitions and properties of Keisler measures, refer-

ring the reader to [4] for more details.

If $\mu \in \mathfrak{M}_x(\mathfrak{C})$ is a global measure and M a small model, we say that μ is M -invariant if $\mu(\phi(x, a) \triangle \phi(x, a')) = 0$ for every formula $\phi(x, y)$ and $a, a' \in \mathfrak{C}$ having the same type over M . Such a measure admits a Borel defining scheme over M : For every formula $\phi(x, y)$, the value $\mu(\phi(x, b))$ depends only on $\text{tp}(b/M)$ and for any Borel $B \subset [0, 1]$, the set $\{p \in S_y(M) : \mu(\phi(x, b)) \in B \text{ for some } b \models p\}$ is a Borel subset of $S_y(M)$.

Let $\mu_x \in \mathfrak{M}(\mathfrak{C})$ be M -invariant. If $\lambda_y \in \mathfrak{M}(\mathfrak{C})$ is any measure, then we can define the *invariant extension* of μ_x over λ_y , denoted $\mu_x \otimes \lambda_y$. It is a measure in the two variables x, y defined in the following way. Let $\phi(x, y) \in L(\mathfrak{C})$. Take a small model N containing M and the parameters of ϕ . Define $\mu_x \otimes \lambda_y(\phi(x, y)) = \int f(p) d\lambda_y$, the integral ranging over $S_y(N)$ where $f(p) = \mu_x(\phi(x, b))$ for $b \in \mathfrak{C}$, $b \models p$ (this function is Borel by Borel definability). It is easy to check that this does not depend on the choice of N .

If λ_y is also invariant, we can also form the product $\lambda_y \otimes \mu_x$. In general it will not be the case that $\lambda_y \otimes \mu_x = \mu_x \otimes \lambda_y$.

If μ_x is a global M -invariant measure, we define by induction: $\mu_{x_1 \dots x_n}^{(n)}$ by $\mu_{x_1}^{(1)} = \mu_{x_1}$ and $\mu_{x_1 \dots x_{n+1}}^{n+1} = \mu_{x_{n+1}} \otimes \mu_{x_1 \dots x_n}^{(n)}$. We let $\mu_{x_1 x_2 \dots}^{(\omega)}$ be the union and call it the *Morley sequence* of μ_x .

Special cases of M -invariant measures include definable and finitely satisfiable measures. A global measure μ_x is *definable* over M if it is M -invariant and for every formula $\phi(x, y)$ and open interval $I \subset [0, 1]$ the set $\{p \in S_y(M) : \mu(\phi(x, b)) \in I \text{ for some } b \models p\}$ is open in $S_y(M)$. The measure μ is *finitely satisfiable* in M if $\mu(\phi(x, b)) > 0$ implies that $\phi(x, b)$ is satisfied in M . Equivalently, any weakly random type for μ is finitely satisfiable in M .

Lemma 1.2. *Let $\mu \in \mathfrak{M}_x(\mathfrak{C})$ be definable over M , and $p(x) \in S_x(\mathfrak{C})$ be weakly random for μ . Let $\phi(x_1, \dots, x_n)$ be a formula over \mathfrak{C} . Suppose that $\phi(x_1, \dots, x_n) \in p^{(n)}$. Then $\mu^{(n)}(\phi(x_1, \dots, x_n)) > 0$.*

Proof. We will carry out the proof in the case where μ is definable (over M), which is anyway the case we need. Note that $p^{(m)}$ is M -invariant for all m . The proof of the lemma is by induction on n . For $n = 1$ it is just the definition of weakly random. Assume true for n and we prove for $n + 1$. So suppose $\phi(x_1, \dots, x_n, x_{n+1}) \in p^{(n+1)}$. This means that for (a_1, \dots, a_n) realizing $p^{(n)}|M$, $\phi(a_1, \dots, a_n, x) \in p$. So as p is weakly random for μ , $\mu(\phi(a_1, \dots, a_n, x)) = r > 0$. So as μ is M -invariant, $\text{tp}(a'_1, \dots, a'_n/M) = \text{tp}(a_1, \dots, a_n/M)$ implies

$\mu(\phi(a'_1, \dots, a'_n, x)) = r$ and thus also $r - \epsilon < \mu(\phi(a'_1, \dots, a'_n, x))$ for any small positive ϵ . By definability of μ and compactness there is a formula $\psi(x_1, \dots, x_n) \in tp(a_1, \dots, a_n/A)$ such that $\models \psi(a'_1, \dots, a'_n)$ implies $0 < r - \epsilon < \mu(\phi(a'_1, \dots, a'_n, x))$. By induction hypothesis, $\mu^{(n)}(\psi(x_1, \dots, x_n)) > 0$. So by definition of $\mu^{(n+1)}$ we have that $\mu^{(n+1)}(\phi(x_1, \dots, x_n, x_{n+1})) > 0$ as required. \square

A measure μ_{x_1, \dots, x_n} is *symmetric* if for any permutation σ of $\{1, \dots, n\}$ and any formula $\phi(x_1, \dots, x_n)$, we have $\mu(\phi(x_1, \dots, x_n)) = \mu(\phi(x_{\sigma.1}, \dots, x_{\sigma.n}))$. A special case of a symmetric measure is given by powers of a generically stable measure as we recall now. The following is Theorem 3.2 of [4]:

Fact 1.3. *Let μ_x be a global M -invariant measure. Then the following are equivalent:*

1. μ_x is both definable and finitely satisfiable (necessarily over M),
2. $\mu_{x_1, \dots, x_n}^{(n)}|_M$ is symmetric for all $n < \omega$,
3. for any global M -invariant Keisler measure λ_y , $\mu_x \otimes \lambda_y = \lambda_y \otimes \mu_x$,
4. μ commutes with itself: $\mu_x \otimes \mu_y = \mu_y \otimes \mu_x$.

If μ_x satisfies one of those properties, we say it is generically stable.

If $\mu \in \mathfrak{M}_x(A)$ and D is a definable set such that $\mu(D) > 0$, we can consider the *localisation* of μ at D which is a Keisler measure μ_D over A defined by $\mu_D(X) = \mu(X \cap D)/\mu(D)$ for any definable set X .

We will use the notation $Fr(\theta(x), x_1, \dots, x_n)$ to mean

$$\frac{1}{n} |\{i \in \{1, \dots, n\} : \models \theta(x_i)\}|.$$

The following is a special case of Lemma 3.4. of [4].

Proposition 1.4. *Let $\phi(x, y)$ be a formula over M and fix $r \in (0, 1)$ and $\epsilon > 0$.*

Then there is n such that for any symmetric measure $\mu_{x_1, \dots, x_{2n}}$, we have

$$\mu_{x_1, \dots, x_{2n}}(\exists y (|Fr(\phi(x, y), x_1, \dots, x_n) - Fr(\phi(x, y), x_{n+1}, \dots, x_{2n})| > r)) \leq \epsilon.$$

2 Main result

Proposition 2.1. *Let μ_x be a global generically stable measure. Let $\phi(x, y)$ be any formula in $L(\mathfrak{C})$. Suppose that $\mu(\phi(x, b)) = 0$ for all $b \in \mathfrak{C}$. Then there is n such that $\mu^{(n)}(\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$.*

Moreover, n depends only on $\phi(x, y)$ and not on μ .

Proof. Let μ_x be a global generically stable measure and M a small model over which $\phi(x, y)$ is defined and such that μ_x is M -invariant. Assume that $\mu(\phi(x, b)) = 0$ for all $b \in \mathfrak{C}$. For any k , define

$$W_k = \{(x_1, \dots, x_n) : \exists y(\wedge_{i=1..k} \phi(x_i, y))\}.$$

This is a definable set. We want to show that $\mu^{(n)}(W_n) = 0$ for n big enough. Assume for a contradiction that this is not the case.

Let n be given by Proposition 1.4 for $r = 1/2$ and $\epsilon = 1/2$. Consider the measure $\lambda_{x_1, \dots, x_{2n}}$ over M defined as being equal to $\mu^{(2n)}$ localised on the set W_{2n} (by our assumption, this is well defined). As the measure $\mu^{(2n)}$ is symmetric and the set W_{2n} is symmetric in the $2n$ variables, the measure λ is symmetric. Let $\chi(x_1, \dots, x_{2n})$ be the formula “ $(x_1, \dots, x_{2n}) \in W_{2n} \wedge \forall y(|Fr(\phi(x, y), x_1, \dots, x_n) - Fr(\phi(x, y), x_{n+1}, \dots, x_{2n})| \leq 1/2)$ ”. By definition of n , we have $\lambda(\exists y(|Fr(\phi(x, y), x_1, \dots, x_n) - Fr(\phi(x, y), x_{n+1}, \dots, x_{2n})| > 1/2)) \leq 1/2$. Therefore $\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) > 0$.

As μ is M -invariant, we can write

$$\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) = \int_{q \in S_{x_1, \dots, x_n}(M)} \mu^{(n)}(\chi(q, x_{n+1}, \dots, x_{2n})) d\mu^{(n)},$$

where $\mu^{(n)}(\chi(q, x_{n+1}, \dots, x_{2n}))$ stands for $\mu^{(n)}(\chi(a_1, \dots, a_n, x_{n+1}, \dots, x_{2n}))$ for some (any) realization (a_1, \dots, a_n) of q . As $\mu^{(2n)}(\chi(x_1, \dots, x_{2n})) > 0$, there is $q \in S_{x_1, \dots, x_n}$ such that

(*) $\mu^{(n)}(\chi(q, x_{n+1}, \dots, x_{2n})) > 0$.

Fix some $(a_1, \dots, a_n) \models q$. By (*), we have $(a_1, \dots, a_n) \in W_n$. So let $b \in \mathfrak{C}$ such that $\models \bigwedge_{i=1..n} \phi(a_i, b)$. Again by (*), we can find some (a_{n+1}, \dots, a_{2n}) weakly random for $\mu^{(n)}$ over $Ma_1 \dots a_n b$ and such that

(**) $\models \chi(a_1, \dots, a_n, a_{n+1}, \dots, a_{2n})$.

In particular, for $j = n+1, \dots, 2n$, a_j is weakly random for μ over Mb hence $\models \neg \phi(a_j, b)$. But then $|Fr(\phi(x, b); a_1, \dots, a_n) - Fr(\phi(x, b); a_{n+1}, \dots, a_{2n})| = 1$. This contradicts (**). \square

Remark 2.2. The proof above adapts to showing the following generalization: Let μ_x be a global generically stable measure, $\phi(x, y)$ a formula in $L(\mathfrak{C})$. Let $\Sigma(x)$ be the partial type (over the parameters in ϕ together with a small model over which μ is definable) defining $\{b : \mu(\phi(x, b)) = 0\}$. Then for some n : $\mu^{(n)}(\exists y(\Sigma(y) \wedge \phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$.

3 Generics in *fsg* groups

Let G be a definable group, without loss defined over \emptyset . We call a definable subset X of G left (right) generic if finitely many left (right) translates of X cover G , and a type $p(x) \in S_G(A)$ is left (right) generic if every formula in p is. We originally defined ([2]) G to have “finitely satisfiable generics”, or to be *fsg*, if there is some global complete type $p(x) \in S_G(\mathfrak{C})$ of G every left G -translate of which is finitely satisfiable in some fixed small model M .

The following summarizes the situation, where the reader is referred to Proposition 4.2 of [2] for (i) and Theorem 7.7 of [3] and Theorem 4.3 of [4] for (ii), (iii), and (iv).

Fact 3.1. *Suppose G is an *fsg* group. Then*

- (i) *A definable subset X of G is left generic iff it is right generic, and the family of nongeneric definable sets is a (proper) ideal of the Boolean algebra of definable subsets of G ,*
- (ii) *There is a left G -invariant Keisler measure $\mu \in \mathfrak{M}_G(\mathfrak{C})$ which is generically stable,*
- (iii) *Moreover μ from (ii) is the unique left G -invariant global Keisler measure on G as well as the unique right G -invariant global Keisler measure on G ,*
- (iv) *Moreover μ from (ii) is generic in the sense that for any definable set X , $\mu(X) > 0$ iff X is generic.*

Remember that a definable set X (or rather a formula $\phi(x, b)$ defining it) forks over a set A if $\phi(x, b)$ implies a finite disjunction of formulas $\psi(x, c)$ each of which divide over A , and $\psi(x, c)$ is said to divide over A if for some A -indiscernible sequence $(c_i : i < \omega)$ with $c_0 = c$, $\{\phi(x, c_i) : i < \omega\}$ is inconsistent.

Proposition 3.2. *Suppose G is *fsg* and $X \subseteq G$ a definable set. Then X is generic if and only if for all $g \in X$, $g \cdot X$ does not fork over \emptyset (if and only if for all $g \in G$, $X \cdot g$ does not fork over \emptyset).*

Proof. Left to right: It suffices to prove that any generic definable set X does not fork over \emptyset , and as the set of nongenerics forms an ideal it is enough to prove that any generic definable set does not divide over \emptyset . This is carried out in (the proof of) Proposition 5.12 of [3].

Right to left: Assume that X is nongeneric. We will prove that for some $g \in G$, $g \cdot X$ divides over \emptyset (so also forks over \emptyset).

Let μ_x be the generically stable G -invariant global Keisler measure given by Fact 3.1. Let M_0 be a small model such that μ does not fork over M_0 (namely, as μ is generic, every generic formula does not fork over M_0) and X is definable over M_0 . Let $\phi(x, y)$ denote the formula defining $\{(x, y) \in G \times G : y \in x \cdot X\}$. So ϕ has additional (suppressed) parameters from M_0 . Note that for $b \in G$, $\phi(x, b)$ defines the set $b \cdot X^{-1}$. As X is nongeneric, so is X^{-1} so also $b \cdot X^{-1}$ for all $b \in G$. Hence, as μ is generic, $\mu(\phi(x, b)) = 0$ for all b . By Proposition 2.1, for some n $\mu^{(n)}(\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y))) = 0$. Let p be any weakly random type for μ (which in this case amounts to a global generic type, which note is M_0 -invariant). So by Lemma 1.2 the formula $\exists y(\phi(x_1, y) \wedge \dots \wedge \phi(x_n, y)) \notin p^{(n)}$. Let (a_1, \dots, a_n) realize $p^{(n)}|_{M_0}$. Then (a_1, \dots, a_n) extends to an M_0 -indiscernible sequence $(a_i : i = 1, 2, \dots)$, a Morley sequence in p over M_0 , and $\models \neg \exists y(\phi(a_1, y) \wedge \dots \wedge \phi(a_n, y))$. So in particular $\{\phi(a_i, y) : i = 1, 2, \dots\}$ is inconsistent. Hence the formula $\phi(a_i, y)$ divides over M_0 , so also divides over \emptyset . But $\phi(a_1, y)$ defines the set $a_1 \cdot X$, so $a_1 \cdot X$ divides over \emptyset as required. \square

Recall that we called a global type $p(x)$ of a \emptyset -definable group G , left f -generic if every left G -translate of p does not fork over \emptyset .

We conclude the following (answering positively Problem 5.5 from [3] as well as strengthening Lemma 4.14 of [1]):

Corollary 3.3. *Suppose G is fsg and $p(x) \in S_G(\mathfrak{C})$. Then the following are equivalent:*

- (i) p is generic,
- (ii) p is left (right) f -generic,
- (iii) (Left or right) $\text{Stab}(p)$ has bounded index in G (where $\text{left Stab}(p) = \{g \in G : g \cdot p = p\}$).

Proof. The equivalence of (i) and (ii) is given by Proposition 3.2 and the definitions. We know from [2], Corollary 4.3, that if p is generic then $\text{Stab}(p)$ is precisely G^{00} . Now suppose that p is nongeneric. Hence there is a definable

set $X \in p$ such that X is nongeneric. Let M be a small model over which X is defined. Note that the *fs* property is invariant under naming parameters. Hence G is *fs* in $Th(\mathfrak{C}, m)_{m \in M}$. By Proposition 3.2 (as well as what is proved in “Right to left” there), for some $g \in G$, $g \cdot X$ divides over M . As X is defined over M this means that there is an M -indiscernible sequence $(g_\alpha : \alpha < \bar{\kappa})$ (where $\bar{\kappa}$ is the cardinality of the monster model) and some n such that $g_{\alpha_1} \cdot X \cap \dots \cap g_{\alpha_n} \cdot X = \emptyset$ whenever $\alpha_1 < \dots < \alpha_n$. This clearly implies that among $\{g_\alpha \cdot p : \alpha < \bar{\kappa}\}$, there are $\bar{\kappa}$ many types, whereby $Stab(p)$ has unbounded index. \square

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